One-dimensional impenetrable anyons in thermal equilibrium: II. Determinant representation

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Abstract

We have obtained a determinant representation for the time- and temperature-dependent field–field correlation function of the impenetrable Lieb–Liniger gas of anyons through direct summation of the form factors. In the static case, the obtained results are shown to be equivalent to those that follow from the anyonic generalization of Lenard's formula.

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1. Introduction and statement of results

This is the second paper in a series that provides a comprehensive treatment of the properties of temperature-dependent correlation functions of one-dimensional (1D) impenetrable free anyons, based on the methods developed for impenetrable bosons [1]. The anyonic model considered in this work can be viewed as a generalization to an arbitrary statistics parameter κ of the model of impenetrable bosons obtained from the Bose gas with repulsive δ-function interaction [1, 2] in the limit of infinitely large coupling constant (for other anyonic extensions of well-known models see [3–5]). This model, which we call the Lieb–Liniger gas of anyons, was formulated in this form by Kundu [6], clarified in [7, 8] and further studied in [9–14]. In the bosonic case, the first step in the analysis of the correlation functions is the derivation of the Fredholm-determinant representation for these functions [15]. With the help of the determinant representation, a classical integrable system characterizing the correlation functions can be constructed as in [16, 17], leading to the short-distance and low-density expansions of the

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correlators. The large-distance asymptotics are then obtained by the inverse scattering method for the integrable system and the solution of the associated matrix Riemann–Hilbert problem [18]. These results will be presented in future publications. The purpose of this work is to derive the Fredholm-determinant representation for the temperature-dependent correlation functions in the case of anyons, and to prove the equivalence of this representation with the anyonic generalization of Lenard's formula [19].

The results of this work can be summarized as follows. One defines free propagators

$$e(\lambda|t, x) = e^{it\lambda^2 - ix\lambda}, \qquad G(t, x) = \int_{-\infty}^{\infty} e(\lambda|t, x) \, d\lambda, \tag{1}$$

and the function

$$E(\mu|t, x) = \text{P.V.} \int_{-\infty}^{\infty} d\lambda \frac{e(\lambda|t, x)}{\lambda - \mu} + e(\mu|t, x)\pi \tan\left(\frac{\pi\kappa}{2}\right), \tag{2}$$

where P.V. denotes the Cauchy principal value. In terms of these functions, the time- and temperature-dependent field–field correlator of impenetrable 1D anyons is

$$\left\langle \Psi_A(x_2, t_2)\Psi_A^\dagger(x_1, t_1)\right\rangle_T = e^{iht_{21}}\left(\frac{1}{2\pi}G(t_{12}, x_{12}) + \frac{\partial}{\partial\alpha}\right)\det(1 + \hat{V}_T)\bigg|_{\alpha=0}, \tag{3}$$

where $x_{ab} = x_a - x_b$, $t_{ab} = t_a - t_b$, $a, b = 1, 2$, and $\det(1 + \hat{V}_T)$ is the Fredholm determinant of the integral operator with the kernel

$$V_T(\lambda, \mu) = \cos^2(\pi\kappa/2)\exp\left\{-\frac{i}{2}t_{12}(\lambda^2 + \mu^2) + \frac{i}{2}x_{12}(\lambda + \mu)\right\}\sqrt{\vartheta(\lambda)\vartheta(\mu)}$$

$$\times\left[\frac{E(\lambda|t_{12}, x_{12}) - E(\mu|t_{12}, x_{12})}{\pi^2(\lambda - \mu)} - \frac{\alpha}{2\pi^3}E(\lambda|t_{12}, x_{12})E(\mu|t_{12}, x_{12})\right], \tag{4}$$

which acts on an arbitrary function $f(\mu)$ as

$$(V_T f)(\lambda) = \int_{-\infty}^{\infty} V_T(\lambda, \mu)f(\mu) \, d\mu. \tag{5}$$

In equation (4), $\vartheta(\lambda) \equiv \vartheta(\lambda, T, h)$ is the Fermi-distribution function at temperature T and chemical potential h,

$$\vartheta(\lambda, T, h) = \frac{1}{1 + e^{(\lambda^2 - h)/T}}. \tag{6}$$

Introducing integral operators $\hat{K}_T$ and $\hat{A}_T^\pm$ which act on the entire real axis and have kernels

$$K_T(\lambda, \mu) = \sqrt{\vartheta(\lambda)}\frac{\sin x(\lambda - \mu)}{\lambda - \mu}\sqrt{\vartheta(\mu)}, \tag{7}$$

and

$$A_T^\pm(\lambda, \mu) = \sqrt{\vartheta(\lambda)}\, e^{\mp ix(\lambda + \mu)}\sqrt{\vartheta(\mu)}. \tag{8}$$

we obtain the static, i.e. equal-time, correlators as

$$\left\langle \Psi_A^\dagger(x)\Psi_A(-x)\right\rangle_T = \frac{1}{2\pi}\text{Tr}\left[(1 - \gamma\hat{K}_T)^{-1}\hat{A}_T^+\right]\det(1 - \gamma\hat{K}_T)|_{\gamma=(1+e^{+i\pi\kappa})/\pi}, \tag{9}$$

and

$$\left\langle \Psi_A^\dagger(-x)\Psi_A(x)\right\rangle_T = \frac{1}{2\pi}\text{Tr}\left[(1 - \gamma\hat{K}_T)^{-1}\hat{A}_T^-\right]\det(1 - \gamma\hat{K}_T)|_{\gamma=(1+e^{-i\pi\kappa})/\pi}. \tag{10}$$

Here $\text{Tr}[f(x, y)] \equiv \int f(x, x) \, dx$, and due to the nonconservation of parity the correlator $\left\langle \Psi_A^\dagger(x)\Psi_A(-x)\right\rangle_T$ is different from $\left\langle \Psi_A^\dagger(-x)\Psi_A(x)\right\rangle_T$.

The paper is organized as follows. Section 2 introduces the Lieb–Liniger gas of anyons and presents the Bethe Ansatz eigenfunctions, Bethe equations, the ground state and the thermodynamics of anyons in the impenetrable limit. In section 3, we compute the form factors and express the field correlator as a Fredholm determinant. Section 4 presents the proof of the equivalence of equations (9) and (10) to the anyonic version of Lenard's formula [19]. Some technical details of the calculations are relegated to the appendices.

2. The Lieb–Liniger gas of impenetrable anyons

The second-quantized Hamiltonian of the Lieb–Liniger gas of 1D anyons is

$$H = \int_{-L/2}^{L/2} dx \left([\partial_x \Psi_A^\dagger(x)][\partial_x \Psi_A(x)] + c \Psi_A^\dagger(x) \Psi_A^\dagger(x) \Psi_A(x) \Psi_A(x) - h \Psi_A^\dagger(x) \Psi_A(x) \right), \quad (11)$$

where $c > 0$ is the coupling constant, L is the length of normalization interval and h is the chemical potential. The canonical Heisenberg fields

$$\Psi_A^\dagger(x, t) = e^{iHt} \Psi_A^\dagger(x) e^{-iHt}, \qquad \Psi_A(x, t) = e^{iHt} \Psi_A(x) e^{-iHt} \qquad (12)$$

obey the anyonic equal-time commutation relations

$$\Psi_A(x_1, t) \Psi_A^\dagger(x_2, t) = e^{-i\pi\kappa\epsilon(x_1-x_2)} \Psi_A^\dagger(x_2, t) \Psi_A(x_1, t) + \delta(x_1 - x_2), \qquad (13)$$

$$\Psi_A^\dagger(x_1, t) \Psi_A^\dagger(x_2, t) = e^{i\pi\kappa\epsilon(x_1-x_2)} \Psi_A^\dagger(x_2, t) \Psi_A^\dagger(x_1, t), \qquad (14)$$

$$\Psi_A(x_1, t) \Psi_A(x_2, t) = e^{i\pi\kappa\epsilon(x_1-x_2)} \Psi_A(x_2, t) \Psi_A(x_1, t), \qquad (15)$$

where κ is the statistics parameter, which we assume to be rational (this is necessary in order for equation (39) to hold), and $\epsilon(x) = x/|x|, \epsilon(0) = 0$. The Fock vacuum is defined as usual by

$$\Psi_A(x)|0\rangle = 0 = \langle 0|\Psi_A^\dagger(x), \qquad \langle 0|0\rangle = 1. \qquad (16)$$

The eigenstates $|\Psi_N\rangle$ of the Hamiltonian are

$$|\Psi_N\rangle = \frac{1}{\sqrt{N!}} \int_{-L/2}^{L/2} dz_1 \cdots \int_{-L/2}^{L/2} dz_N \, \chi_N(z_1, \ldots, z_N|\lambda_1, \ldots, \lambda_N) \Psi_A^\dagger(z_N) \cdots \Psi_A^\dagger(z_1)|0\rangle, \qquad (17)$$

where quantum-mechanical wavefunctions have the property of anyonic exchange statistics:

$$\chi_N(\ldots, z_i, z_{i+1}, \ldots) = e^{i\pi\kappa\epsilon(z_i-z_{i+1})} \chi_N(\ldots, z_{i+1}, z_i, \ldots). \qquad (18)$$

Note that the sign in front of the statistical phase in this expression ($+i\pi\kappa$ or $-i\pi\kappa$) depends on the choice of ordering of the creation operators in the definition of the eigenstates (17). The order of these operators adopted in equation (17) (leading to the phase $+i\pi\kappa$): the particle with the first coordinate z_1 created first, then z_2, etc, is convenient [7] for the subsequent calculation of the form factors.

In this paper, we limit our discussion to the case of infinitely strong interaction, $c \to \infty$, which corresponds to impenetrable anyons. In general, the eigenfunctions χ_N are [8]

$$\chi_N = \frac{e^{+i\frac{\pi\kappa}{2} \sum_{j<k} \epsilon(z_j-z_k)}}{\sqrt{N! \prod_{j>k}[(\lambda_j - \lambda_k)^2 + c'^2]}} \sum_{\pi \in S_N} (-1)^\pi e^{i \sum_{n=1}^N z_n \lambda_{\pi(n)}}$$
$$\times \prod_{j>k} [\lambda_{\pi(j)} - \lambda_{\pi(k)} - ic'\epsilon(z_j - z_k)], \qquad (19)$$

where $c' \equiv c/\cos(\pi\kappa/2)$, and reduce for impenetrable anyons to a simpler form:

$$\chi_N = \frac{\mathrm{e}^{+\mathrm{i}\frac{\pi\kappa}{2}\sum_{j<k}\epsilon(z_j-z_k)}}{\sqrt{N!}} \prod_{j>k} \epsilon(z_j - z_k) \sum_{\pi\in S_N} (-1)^\pi \mathrm{e}^{\mathrm{i}\sum_{n=1}^N z_n\lambda_{\pi(n)}}. \tag{20}$$

Here S_N is the group of permutations of N elements and $(-1)^\pi$ is the sign of the permutation. The energy eigenvalues

$$H|\Psi_N\rangle = E|\Psi_N\rangle$$

are given by the sum of effective single-particle contributions:

$$E = \sum_{i=1}^N \varepsilon(\lambda_j), \qquad \text{with} \qquad \varepsilon(\lambda) = \lambda^2 - h. \tag{21}$$

The individual momenta λ_j depend on the boundary conditions imposed on the wavefunctions. In contrast to particles of integer statistics, wavefunctions of the anyons satisfy different quasi-periodic boundary conditions in their different arguments, the difference resulting from the statistical phase shift $2\pi\kappa$ [7, 8]. In general, the quasi-periodic boundary conditions also include the external phase shift η (we will consider $\eta = 2\pi \times$ rational), so that the boundary conditions on the wavefunctions (20) are:

$$\chi_N(-L/2, z_2, \ldots, z_N) = \mathrm{e}^{-\mathrm{i}\eta}\chi_N(L/2, z_2, \ldots, z_N),$$
$$\chi_N(z_1, -L/2, \ldots, z_N) = \mathrm{e}^{\mathrm{i}(2\pi\kappa-\eta)}\chi_N(z_1, L/2, \ldots, z_N),$$
$$\vdots$$
$$\chi_N(z_1, z_2, \ldots, -L/2) = \mathrm{e}^{\mathrm{i}(2\pi(N-1)\kappa-\eta)}\chi_N(z_1, z_2\ldots, L/2). \tag{22}$$

The difference in the boundary conditions for different arguments of χ_N makes it possible, in general, to impose the condition without the statistical phase shift on any of the arguments z_j. The precise form of the Bethe equations for the momenta λ_j in wavefunction (19) depends on a specific choice of the boundary conditions. The choice (22), in which the first coordinate z_1 does not have the statistical shift under its boundary condition, gives rise to the Bethe equations which include the full statistical contribution $\pi\kappa(N-1)$ to the momentum shift of each of the anyons produced by the $N-1$ other anyons in the system [8]:

$$\mathrm{e}^{\mathrm{i}\lambda_j L} = \mathrm{e}^{\mathrm{i}\bar{\eta}} \prod_{k=1, k\neq j}^N \left(\frac{\lambda_j - \lambda_k + \mathrm{i}c'}{\lambda_j - \lambda_k - \mathrm{i}c'}\right), \tag{23}$$

where $\bar{\eta} = \eta - \pi\kappa(N-1)$. Similarly to the wavefunctions, the general Bethe equations (23) are simplified in the impenetrable limit $c \to \infty$:

$$\mathrm{e}^{\mathrm{i}\lambda_j L} = (-1)^{N-1}\mathrm{e}^{\mathrm{i}\bar{\eta}}. \tag{24}$$

2.1. Structure of the ground state

We assume that the ground state of the gas contains N anyons, and take, for convenience, N to be even, although this does not affect our final results. We denote the momenta of the particles in the ground state as μ_j, where $j = 1, \ldots, N$, and introduce the notation $\{[\cdots]\}$ such that

$$\{[x]\} = \gamma, \qquad \text{if } x = 2\pi \times \text{ integer } + 2\pi\gamma, \quad \gamma \in (-1, 1). \tag{25}$$

The Bethe equations (24) then give the momenta μ_j:

$$\mu_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right) + \frac{2\pi\delta}{L}, \qquad j = 1, \ldots, N_0, \tag{26}$$

where $\delta = \{[\overline{\eta}]\}$. In the thermodynamic limit $L \to \infty$, $N \to \infty$, $N/L = D$, momenta of the particles fill densely the Fermi sea $[-q, q]$, where $q = \sqrt{h}$ is the Fermi momentum and the gas density is $D = q/\pi$.

2.2. Thermodynamics

The thermodynamics of the Lieb–Liniger anyonic gas was considered in [10, 11]. Similarly to the structure of the ground state, all local thermodynamic characteristics in the case of impenetrable anyons are equivalent to those of the free fermions. At non-vanishing temperature T, the quasiparticle distribution is given by the Fermi weight (6), and the density and energy are

$$D = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vartheta(\lambda, h, T) \, d\lambda, \qquad E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda^2 \vartheta(\lambda, h, T) \, d\lambda. \tag{27}$$

The density increases monotonically as a function of the chemical potential h. At $T = 0$, we have $D = 0$ for $h \leqslant 0$, and $0 < D < \infty$ if $0 < h < \infty$. At non-vanishing temperature, the density is zero for $h = -\infty$ and monotonically increases with h for $-\infty < h < \infty$.

3. The time-dependent field–field correlator

In our previous paper [19], we have derived the anyonic generalization of the Lenard formula, which for impenetrable free anyons, is an expansion of the anyonic reduced density matrices in terms of the reduced density matrices of free fermions. In the simplest case, the correlator

$$\left(x_1 \big| \rho_1^a \big| x_2\right) = \left\langle \Psi_A^\dagger(x_2) \Psi_A(x_1) \right\rangle_T \tag{28}$$

is the first Fredholm minor of an integral operator, whose kernel is the Fourier transform of the Fermi weight (6). In this section, we obtain the time dependent generalization of this result. Our approach will be based on the following considerations. We start with the zero-temperature field correlator

$$\left\langle \Psi_A(x_2, t_2) \Psi_A^\dagger(x_1, t_1) \right\rangle_N = \frac{\langle \Psi(\mu_1, \ldots, \mu_N) | \Psi_A(x_2, t_2) \Psi_A^\dagger(x_1, t_1) | \Psi(\mu_1, \ldots, \mu_N) \rangle}{\langle \Psi(\mu_1, \ldots, \mu_N) | \Psi(\mu_1, \ldots, \mu_N) \rangle}, \tag{29}$$

where the wavefunctions are taken to be normalized as

$$\langle \Psi(\mu_1, \ldots, \mu_N) | \Psi(\mu_1, \ldots, \mu_N) \rangle = L^N, \tag{30}$$

and $\mu_1, \ldots, \mu_N$ are the momenta in the ground state (26). Using the resolution of identity for the Hilbert space of $N + 1$ particles

$$\mathbf{1} = \sum_{\text{all}\{\lambda\}_{N+1}} \frac{|\Psi(\lambda_1, \ldots, \lambda_{N+1})\rangle\langle\Psi(\lambda_1, \ldots, \lambda_{N+1})|}{\langle \Psi(\lambda_1, \ldots, \lambda_{N+1}) | \Psi(\lambda_1, \ldots, \lambda_{N+1}) \rangle}, \tag{31}$$

where, according to (30),

$$\langle \Psi(\lambda_1, \ldots, \lambda_{N+1}) | \Psi(\lambda_1, \ldots, \lambda_{N+1}) \rangle = L^{N+1},$$

and the sum is over all possible solutions of the Bethe equations with $N + 1$ particles, we have

$$\left\langle \Psi_A(x_2, t_2) \Psi_A^\dagger(x_1, t_1) \right\rangle_N = \frac{1}{L^{2N+1}} \sum_{\text{all}\{\lambda\}_{N+1}} \langle \Psi_N(\{\mu\}) | \Psi_A(x_2, t_2) | \Psi_{N+1}(\{\lambda\}) \rangle$$

$$\times \left\langle \Psi_{N+1}(\{\lambda\}) | \Psi_A^\dagger(x_1, t_1) | \Psi_N(\{\mu\}) \right\rangle. \tag{32}$$

Defining the form factors

$$F_{N+1,N}(x,t) = \langle \Psi_{N+1}(\{\lambda\})|\Psi_A^\dagger(x,t)|\Psi_N(\{\mu\})\rangle,$$
$$F_{N+1,N}^*(x,t) = \langle \Psi_N(\{\mu\})|\Psi_A(x,t)|\Psi_{N+1}(\{\lambda\})\rangle, \tag{33}$$

we can rewrite equation (32) as

$$\langle \Psi_A(x_2,t_2)\Psi_A^\dagger(x_1,t_1)\rangle_N = \frac{1}{L^{2N+1}} \sum_{\text{all}\{\lambda\}_{N+1}} F_{N+1,N}^*(x_2,t_2)F_{N+1,N}(x_1,t_1). \tag{34}$$

Equation (34) means that in order to find the dynamic field correlator, we need to compute the form factors and sum over all of them. After the summation, one can take the thermodynamic limit. In general, such a summation of form factors is extremely difficult. The main simplification which makes it possible to perform this summation in the model of anyons we consider here, is the fact that, similarly to the problem of impenetrable bosons [1, 15], the local thermodynamic properties of particles with δ-function interaction are identical to those of free fermions regardless of the actual exchange statistics. Finally, the finite-temperature correlator can be obtained from the zero-temperature result using the standard argument developed for the Bose gas (see, e.g., appendix 13.1 of [1]), which is also applicable in the case of anyons.

3.1. Form factors

As a first step in carrying out the program outlined above, we compute the form factors. In definition (33) of the form factors, the eigenstates $|\Psi_N(\{\mu\})\rangle$, $|\Psi_{N+1}(\{\lambda\})\rangle$ have, respectively, N and $N+1$ particles. Although the set $\{\mu\}$ represents in (33) momenta in the ground state of N particles, our calculation below is valid also when $|\Psi_N(\{\mu\})\rangle$ is not the ground state. As before, we assume for convenience that N is even. We denote by $\{\mu_j\}$ the momenta of the anyons in the N-particle eigenstate, and by $\{\lambda_j\}$ the momenta in the $N+1$ eigenstate.

Using the definition (17) for the eigenstates with N and $N+1$ anyons

$$|\Psi_N(\{\mu\})\rangle = \frac{1}{\sqrt{N!}} \int d^N z \, \chi_N(z_1,\ldots,z_N|\{\mu\})\Psi_A^\dagger(z_N)\ldots\Psi_A^\dagger(z_1)|0\rangle,$$

$$\langle \Psi_{N+1}(\{\lambda\})| = \frac{1}{\sqrt{N+1!}} \int d^{N+1} y \, \langle 0|\Psi_A(y_1)\ldots\Psi_A(y_{N+1})\chi_{N+1}^*(y_1,\ldots,y_{N+1}|\{\lambda\})$$

one can write the form factor as

$$F_{N+1,N}(x,0) = \frac{1}{\sqrt{(N+1)!N!}} \int d^{N+1}y \, d^N z \, \chi_{N+1}^*(y_1,\ldots,y_{N+1}|\{\lambda\})\chi_N(z_1,\ldots,z_N|\{\mu\})\cdot$$
$$\langle 0|\Psi_A(y_1)\cdots\Psi_A(y_{N+1})\Psi_A^\dagger(x)\Psi_A^\dagger(z_N)\cdots\Psi_A^\dagger(z_1)|0\rangle. \tag{35}$$

A direct application of the anyonic commutation relation (13) and equation (16) described in more detail in appendix A, reduces this expression to

$$F_{N+1,N}(x,0) = \langle \Psi_{N+1}|\Psi_A^\dagger(x)|\Psi_N\rangle$$
$$= \sqrt{N+1} \int d^N z \, \chi_{N+1}^*(z_1,\ldots,z_N,x|\{\lambda\})\chi_N(z_1,\ldots,z_N|\{\mu\}). \tag{36}$$

An important feature of equation (36) is that the order of the creation operators chosen in equation (17) makes the 'free' coordinate x in (36) the last argument of the wavefunction χ_{N+1}. This ensures that both wavefunctions, χ_N and χ_{N+1}, have the same phase shifts (22) at the boundary of the normalization interval in all other variables z_j. Since these phase shifts are canceled in equation (36), the expression under the integrals over z_j is periodic in each of

the variables [7]. This feature is the necessary consistency condition for the Hilbert spaces of anyon wavefunctions with different numbers of particles, and is important in what follows for the appropriate calculation of the form factors (36).

The sets of momenta $\{\mu_j\}$ and $\{\lambda_j\}$ in the wavefunctions χ_N and χ_{N+1} in (36) are determined by the Bethe equations (24) as

$$\mu_j = \frac{2\pi}{L}\left(m_j + \frac{1}{2}\right) + \frac{2\pi\delta}{L}, \qquad \delta = \{[\eta - \pi\kappa(N-1)]\}, \quad j = 1, \ldots, N, m_j \in \mathbb{Z}, \quad (37)$$

$$\lambda_j = \frac{2\pi}{L}n_j + \frac{2\pi\delta'}{L}, \qquad \delta' = \{[\eta - \pi\kappa N]\}, \quad j = 1, \ldots, N+1, n_j \in \mathbb{Z}. \quad (38)$$

These equations show that

$$\lambda_j - \mu_k = \frac{2\pi}{L}\left(l - \frac{\kappa+1}{2}\right), \qquad l \in \mathbb{Z}, \quad (39)$$

which means that λ_j and μ_k never coincide except in the trivial case $\kappa = 1$, when we have a gas of non-interacting fermions. In all other situations, λ_j and μ_k are different. This difference between them comes from the phase shift due to the hard-core condition on the added particle described by the factor $1/2$ in (39), and the extra anyonic statistical phase added to the anyon system together with the particle [7]. This difference between λ_j and μ_k plays an important role in the following calculations. Using the identity

$$e^{+i\frac{\pi\kappa}{2}\epsilon(x-y)}\epsilon(y-x) = \cos(\pi\kappa/2)\epsilon(y-x) - i\sin(\pi\kappa/2), \quad (40)$$

we can rewrite the anyonic wavefunction (20) as

$$\chi_N(z_1, \ldots, z_N|\{\mu\}) = \frac{\prod_{j>k}[\cos(\pi\kappa/2)\epsilon(z_j - z_k) - i\sin(\pi\kappa/2)]}{\sqrt{N!}} \sum_{\pi \in S_N} (-1)^\pi e^{i\sum_{n=1}^N z_n \mu_{\pi(n)}}. \quad (41)$$

Using this expression for both of the wavefunctions in (36) we obtain

$$F_{N+1,N}(x, 0) = \frac{1}{N!} \sum_{\pi \in S_{N+1}} \sum_{\sigma \in S_N} (-1)^{\pi+\sigma} e^{-ix\lambda_{\pi(n+1)}} \int_{-L/2}^{L/2} \prod_{n=1}^N dz_n [\cos(\pi\kappa/2)\epsilon(x - z_n)$$

$$+ i\sin(\pi\kappa/2)] e^{-i\sum_{n=1}^N z_n(\lambda_{\pi(n)} - \mu_{\sigma(n)})}. \quad (42)$$

Integration by parts in this equation produces the boundary terms in the following form:

$$\frac{e^{-iz_n(\lambda_{\pi(n)} - \mu_{\sigma(n)})}}{-i(\lambda_{\pi(n)} - \mu_{\sigma(n)})}(\cos(\pi\kappa/2)\epsilon(x - z_n) + i\sin(\pi\kappa/2))\Bigg|_{z_n=-L/2}^{z_n=L/2}$$

$$= \frac{e^{-i\frac{\pi\kappa}{2}}e^{-i\frac{L}{2}(\lambda_{\pi(n)} - \mu_{\sigma(n)})}}{i(\lambda_{\pi(n)} - \mu_{\sigma(n)})}(1 + e^{+i\pi\kappa}e^{iL(\lambda_{\pi(n)} - \mu_{\sigma(n)})}). \quad (43)$$

All these terms vanish due to equation (39). Then, using the relation

$$\frac{d\epsilon(x - z_n)}{dz_n} = -2\delta(x - z_n), \quad (44)$$

we obtain the following expression for the form factors:

$$F_{N+1,N}(x, 0) = \frac{[2i\cos(\pi\kappa/2)]^N}{N!} \exp\left\{ix\left[\sum_{j=1}^N \mu_j - \sum_{j=1}^{N+1} \lambda_j\right]\right\}$$

$$\times \sum_{\pi \in S_{N+1}} \sum_{\sigma \in S_N} (-1)^{\pi+\sigma} \prod_{j=1}^N \frac{1}{\lambda_{\pi(j)} - \mu_{\sigma(j)}}. \quad (45)$$

This expression differs from the corresponding result for impenetrable bosons [1, 15] by the spectrum of the momenta which now include the statistical shift, and by the overall $[\cos(\pi\kappa/2)]^N$ factor. For $\kappa = 0$, both differences disappear, and equation (45) reproduces, as should be, the case of the Bose gas. We transform this equation following the corresponding steps for bosons [1, 15]. One can see directly that the sums over permutations in (45) can be written in the form of a determinant:

$$\frac{1}{N!} \sum_{\pi \in S_{N+1}} \sum_{\sigma \in S_N} (-1)^{\pi+\sigma} \prod_{j=1}^{N} \frac{1}{\lambda_{\pi(j)} - \mu_{\sigma(j)}} = \left(1 + \frac{\partial}{\partial\alpha}\right) \det_N \left(M_{jk}^{\alpha}\right)\Big|_{\alpha=0}, \tag{46}$$

with

$$M_{jk}^{\alpha} = \frac{1}{\lambda_j - \mu_k} - \frac{\alpha}{\lambda_{N+1} - \mu_k}, \qquad j, k = 1, \ldots, N, \tag{47}$$

reducing equation (45) to

$$F_{N+1,N}(x, 0) = (2\mathrm{i}\cos(\pi\kappa/2))^N \exp\left\{\mathrm{i}x\left[\sum_{j=1}^{N}\mu_j - \sum_{j=1}^{N+1}\lambda_j\right]\right\} \left(1 + \frac{\partial}{\partial\alpha}\right) \det_N \left(M_{jk}^{\alpha}\right)\Big|_{\alpha=0}. \tag{48}$$

The determinant part of this equation can also be written as

$$\left(1 + \frac{\partial}{\partial\alpha}\right) \det_N \left(M_{jk}^{\alpha}\right)\Big|_{\alpha=0} = \sum_{\pi \in S_{N+1}} (-1)^{\pi} \prod_{j=1}^{N} \frac{1}{\lambda_{\pi(j)} - \mu_j}, \tag{49}$$

as one can see directly from the LHS of (46) by noting that due to the permutations π of λ_j, all permutations of μ_j give identical contributions to the sum over $\pi \in S_{N+1}$.

Alternatively, one can introduce a fictitious momentum μ_{N+1}, and obtain the following representation [20] of the form factor in terms of this momentum:

$$F_{N+1,N}(x, 0) = (2\mathrm{i}\cos(\pi\kappa/2))^N$$
$$\times \exp\left\{\mathrm{i}x\left[\sum_{j=1}^{N}\mu_j - \sum_{j=1}^{N+1}\lambda_j\right]\right\} \lim_{\mu_{N+1}\to\infty} \left[-\mu_{N+1}\det_{N+1}\left(\frac{1}{\lambda_j - \mu_k}\right)\right], \tag{50}$$

where $\det_{N+1}(a_{jk})$ is the determinant of the $(N+1) \times (N+1)$ matrix with elements a_{jk}. We will not be using this representation explicitly below.

The time-dependent form factors can be obtained from the timeless form (48) using the following simple relations:

$$\mathrm{e}^{-\mathrm{i}Ht}|\Psi_N(\{\mu\})\rangle = \mathrm{e}^{-\mathrm{i}t\sum_{j=1}^{N}(\mu_j^2 - h)}|\Psi_N(\{\mu\})\rangle, \tag{51}$$

and

$$\langle\Psi_N(\{\lambda\})|\,\mathrm{e}^{\mathrm{i}Ht} = \mathrm{e}^{\mathrm{i}t\sum_{j=1}^{N+1}(\lambda_j^2 - h)}\langle\Psi_N(\{\lambda\})|. \tag{52}$$

Combining the exponential factors in these expressions with those in equation (48), we arrive at the final result for the time-dependent form factor:

$$F_{N+1,N}(x, t) = (2\mathrm{i}\cos(\pi\kappa/2))^N \mathrm{e}^{-\mathrm{i}ht} \left(\prod_{i=1}^{N+1} e(\lambda_i|t, x)\right) \left(\prod_{j=1}^{N} e^*(\mu_j|t, x)\right)$$
$$\times \left(1 + \frac{\partial}{\partial\alpha}\right) \det_N \left(M_{jk}^{\alpha}\right)\Big|_{\alpha=0}, \tag{53}$$

where we have introduced the function

$$e(\lambda|t, x) = e^{it\lambda^2 - ix\lambda}, \tag{54}$$

$e^*(\lambda|t, x)$ is its complex conjugate, and M_{jk}^{α} is defined in (47). The form factor of the annihilation operator $\Psi_A(x, t)$ is obtained through complex conjugation

$$\langle \Psi_N(\{\mu\})|\Psi_A(x, t)|\Psi_{N+1}(\{\lambda\})\rangle = F_{N+1,N}^*(x, t). \tag{55}$$

3.2. Summation of the form factors

Using equations (53) and (55), we write the field correlator (34) as a sum over intermediate momenta $\{\lambda\}$:

$$
\langle \Psi_A(x_2, t_2) \Psi_A^\dagger(x_1, t_1) \rangle_N = \sum_{\text{all } \{\lambda\}_{N+1}} \frac{(2\cos(\pi\kappa/2))^{2N}}{L^{2N+1}} e^{iht_{21}} \left(\prod_{i=1}^{N+1} e^*(\lambda_i|t_{21}, x_{21}) \right)
$$

$$
\times \left(\prod_{j=1}^{N} e(\mu_j|t_{21}, x_{21}) \right) \left(1 + \frac{\partial}{\partial\alpha}\right) \det_N \left(M_{jk}^{\alpha} \right)\big|_{\alpha=0}
$$

$$
\times \left(1 + \frac{\partial}{\partial\beta}\right) \det_N \left(M_{jk}^{\beta} \right)\big|_{\beta=0}, \tag{56}
$$

with the notations $x_{ab} = x_a - x_b, t_{ab} = t_a - t_b, a, b = 1, 2$. The matrix M_{jk}^{β} here is the same as (47) with α replaced by β. As was mentioned above, modulo the $[\cos(\pi\kappa/2)]^{2N}$ factors and the spectrum of momenta, equation (56) is identical to the expression for the bosonic field correlators [1, 15]. This means that the summation process over $\{\lambda\}$ is very similar, and we just sketch the derivation here. Since we sum over all momenta $\{\lambda\}$, individual momenta λ_j are equivalent up to permutation. This means that one of the two permutations of $\{\lambda_j\}$ involved in the definition of the two determinants in (56) produces coinciding terms, so that under the sum over $\{\lambda_j\}$, one can replace one of the determinants, e.g., the second one, with

$$(N + 1)! \prod_{j=1}^{N} \frac{1}{\lambda_j - \mu_j}, \tag{57}$$

obtaining

$$
\langle \Psi_A(x_2, t_2) \Psi_A^\dagger(x_1, t_1) \rangle_N
$$

$$
= e^{iht_{21}} \left(\prod_{j=1}^{N} e(\mu_j|t_{21}, x_{21}) \right) \frac{1}{L} \left(\frac{2\cos(\pi\kappa/2)}{L} \right)^{2N} (N + 1)!
$$

$$
\times \sum_{\text{all } \{\lambda\}_{N+1}} \left(e^*(\lambda_{N+1}|t_{21}, x_{21}) + \frac{\partial}{\partial\alpha} \right) \det_N \left(\frac{e^*(\lambda_j|t_{21}, x_{21})}{(\lambda_j - \mu_k)(\lambda_j - \mu_j)} \right.
$$

$$
\left. - \alpha \frac{e^*(\lambda_j|t_{21}, x_{21})}{(\lambda_j - \mu_j)} \frac{e^*(\lambda_{N+1}|t_{21}, x_{21})}{(\lambda_{N+1} - \mu_j)} \right)\Bigg|_{\alpha=0}. \tag{58}
$$

The summation over the momenta $\{\lambda_j\}$ can be done then independently over each λ_j inside the determinant. Also, we transfer the factors $e(\mu_j|t_{21}, x_{21})$ in (58) into the determinant splitting them between the rows and columns, and use the formula

$$\frac{1}{(\lambda_j - \mu_k)} \frac{1}{(\lambda_j - \mu_j)} = \left(\frac{1}{\lambda_j - \mu_j} - \frac{1}{\lambda_j - \mu_k} \right) \frac{1}{\mu_j - \mu_k}. \tag{59}$$

This gives the correlator as

$$\left\langle \Psi_A(x_2, t_2)\Psi_A^\dagger(x_1, t_1)\right\rangle_N = e^{iht_{21}} \left(\frac{1}{2\pi}G_L(t_{12}, x_{12}) + \frac{\partial}{\partial\alpha} \right)$$

$$\times \det_N \Bigg[\delta_{jk}\tilde{E}_L(\mu_k|t_{12}, x_{12})e(\mu_j|t_{21}, x_{21}) + e(\mu_j|t_2, x_2)e^*(\mu_k|t_1, x_1)\cos^2(\pi\kappa/2)$$

$$\times \Bigg(\frac{2(1-\delta_{jk})}{\pi L(\mu_j - \mu_k)}(E_L(\mu_j|t_{12}, x_{12}) - E_L(\mu_k, t_{12}, x_{12}))$$

$$- \frac{\alpha}{L\pi^2}E_L(\mu_j|t_{12}, x_{12})E_L(\mu_k|t_{12}, x_{12}) \Bigg) \Bigg] \Bigg|_{\alpha=0}, \tag{60}$$

where we have defined the functions

$$\frac{1}{2\pi}G_L(t, x) = \frac{1}{L}\sum_\lambda e(\lambda|t, x), \tag{61}$$

$$\frac{1}{2\pi}E_L(\mu_k|t, x) = \frac{1}{L}\sum_\lambda \frac{e(\lambda|t, x)}{\lambda - \mu_k}, \tag{62}$$

$$\tilde{E}_L(\mu_k|t, x) = \frac{4\cos(\pi\kappa/2)^2}{L^2}\sum_\lambda \frac{e(\lambda|t, x)}{(\lambda - \mu_k)^2}, \tag{63}$$

and $\lambda = \frac{2\pi}{L}(\mathbb{Z} + \delta')$ —see (38). Formula (60) is the final expression for the field correlator in the ground state of N anyons on a finite interval with quasi-periodic boundary conditions.

3.3. Thermodynamic limit

In order to obtain the correlator in the thermodynamic limit, we need to compute the large-L limit of the functions (61), (62) and (63). This is done in appendix B with the results

$$G(t, x) \equiv \lim_{L\to\infty} G_L(t, x) = \int_{-\infty}^\infty e(\lambda|t, x)\,\mathrm{d}\lambda, \tag{64}$$

$$E(\mu_k|t, x) \equiv \lim_{L\to\infty} E_L(\mu_k|t, x) = \text{P.V.}\int_{-\infty}^\infty \mathrm{d}\lambda\,\frac{e(\lambda|t, x)}{\lambda - \mu_k} + e(\mu_k|t, x)\pi\tan\left(\frac{\pi\kappa}{2}\right), \tag{65}$$

$$\tilde{E}(\mu_k|t, x) \equiv \lim_{L\to\infty} \tilde{E}_L(\mu_k|t, x) = e(\mu_k|t, x) + \frac{2\cos^2(\pi\kappa/2)}{\pi L}\frac{\partial}{\partial\mu_k}E(\mu_k|t, x). \tag{66}$$

In the thermodynamic limit $L, N \to \infty$ with $D = N/L$ constant, the anyon momenta fill densely the Fermi interval $[-q, q]$, where $q = \sqrt{h}$ and $D = q/\pi$. In this case, the determinant in the correlator (60) can be understood as the Fredholm determinant of an integral operator. Indeed, for an arbitrary integral operator $\hat{V}$, whose action on a function $f(\lambda)$ is defined by

$$(\hat{V}f)(\lambda) = \int_a^b V(\lambda, \mu)f(\mu)\,\mathrm{d}\mu, \tag{67}$$

the associated Fredholm determinant is (see, e.g., [22])

$$\det(1 + \hat{V}) = \lim_{n\to\infty} \begin{vmatrix} 1 + \xi V(\lambda_1, \lambda_1) & \xi V(\lambda_1, \lambda_2) & \cdots & \xi V(\lambda_1, \lambda_n) \\ \xi V(\lambda_2, \lambda_1) & 1 + \xi V(\lambda_2, \lambda_2) & \cdots & \xi V(\lambda_2, \lambda_n) \\ \vdots & \vdots & \ddots & \vdots \\ \xi V(\lambda_n, \lambda_1) & \xi V(\lambda_n, \lambda_2) & \cdots & 1 + \xi V(\lambda_n, \lambda_n) \end{vmatrix}, \tag{68}$$

where $\xi = (b - a)/n$, $\lambda_p - \lambda_{p-1} = \xi$ and $\lambda_0 = a$, $\lambda_n = b$. One can see directly that, in the thermodynamic limit, the determinant part of equation (60) has the same structure with N momenta μ_j separated by $\xi = 2\pi/L$ filling the Fermi interval $[-q, q]$. This means that the correlator can be expressed as

$$\langle \Psi_A(x_2, t_2) \Psi_A^\dagger(x_1, t_1) \rangle = e^{iht_{21}} \left(\frac{1}{2\pi} G(t_{12}, x_{12}) + \frac{\partial}{\partial \alpha} \right) \det(1 + \hat{\tilde{V}}_0)|_{\alpha=0}, \tag{69}$$

where $\hat{\tilde{V}}_0$ acts on an arbitrary function $f(\lambda)$ as

$$(\hat{\tilde{V}}_0 f)(\lambda) = \int_{-q}^{q} \tilde{V}_0(\lambda, \mu) f(\mu) \, d\mu, \tag{70}$$

and

$$\tilde{V}_0(\lambda, \mu) = \cos^2(\pi\kappa/2) e(\lambda | t_2, x_2) e^*(\mu | t_1, x_1)$$
$$\times \left[\frac{E(\lambda | t_{12}, x_{12}) - E(\mu | t_{12}, x_{12})}{\pi^2 (\lambda - \mu)} - \frac{\alpha}{2\pi^3} E(\lambda | t_{12}, x_{12}) E(\mu | t_{12}, x_{12}) \right]. \tag{71}$$

Performing the unitary transformation

$$V_0(\lambda, \mu) = \exp \left\{ -i \frac{(t_1 + t_2)}{2} (\lambda^2 - \mu^2) + i \frac{(x_1 + x_2)}{2} (\lambda - \mu) \right\} \tilde{V}_0(\lambda, \mu), \tag{72}$$

with the property

$$\det(1 + \hat{\tilde{V}}_0) = \det(1 + \hat{V}_0), \tag{73}$$

we transform the kernel $\tilde{V}_0(\lambda, \mu)$ (71) into the symmetric form:

$$V_0(\lambda, \mu) = \cos^2(\pi\kappa/2) \exp \left\{ -\frac{i}{2} t_{12}(\lambda^2 + \mu^2) + \frac{i}{2} x_{12}(\lambda + \mu) \right\}$$
$$\times \left[\frac{E(\lambda | t_{12}, x_{12}) - E(\mu | t_{12}, x_{12})}{\pi^2 (\lambda - \mu)} - \frac{\alpha}{2\pi^3} E(\lambda | t_{12}, x_{12}) E(\mu | t_{12}, x_{12}) \right]. \tag{74}$$

Two observations are in order. First, one can check that the second term in (66) is obtained from the first term in the square bracket of (74) in the limit $\lambda \to \mu$. Second, in the limit $\kappa \to 0$, equation (74) reproduces the known result [1, 15] for impenetrable bosons.

In the static case ($t_1 = t_2$), which is discussed in the following section, the kernel (74) can be simplified further. One needs to distinguish two cases.

- $x_1 > x_2$. In this case,

$$E(\lambda | 0, x_{12}) = -i\pi \, e^{-ix_{12}\lambda} [1 + i \tan(\pi\kappa/2)], \tag{75}$$

and the kernel (74) becomes

$$V_0^+(\lambda, \mu) = -\frac{(1 + e^{+i\pi\kappa})}{\pi} \left(\frac{\sin(x_{12}(\lambda - \mu)/2)}{\lambda - \mu} \right) + \frac{\alpha}{2\pi} e^{+i\pi\kappa} \exp \left\{ -i \frac{x_{12}}{2} (\lambda + \mu) \right\}. \tag{76}$$

- $x_1 < x_2$. In this case,

$$E(\lambda | 0, x_{12}) = i\pi \, e^{-ix_{12}\lambda} [1 - i \tan(\pi\kappa/2)], \tag{77}$$

and

$$V_0^-(\lambda, \mu) = \frac{(1 + e^{-i\pi\kappa})}{\pi} \left(\frac{\sin(x_{12}(\lambda - \mu)/2)}{\lambda - \mu} \right) + \frac{\alpha}{2\pi} e^{-i\pi\kappa} \exp \left\{ -i \frac{x_{12}}{2} (\lambda + \mu) \right\}. \tag{78}$$

We now extend the discussion to the situation of non-vanishing temperature T. The temperature-dependent field correlator is defined as

$$\left\langle \Psi_A(x_2, t_2)\Psi_A^\dagger(x_1, t_2)\right\rangle_T = \frac{\mathrm{Tr}\big(\mathrm{e}^{-H/T}\Psi_A(x_2, t_2)\Psi_A^\dagger(x_1, t_1)\big)}{\mathrm{Tr}\,\mathrm{e}^{-H/T}}. \tag{79}$$

According to the well-known argument developed for the Bose gas [1], this correlator can be found as the mean value over any one of the 'typical' eigenfunctions Ω_T of the Hamiltonian which characterizes the given state of thermal equilibrium:

$$\frac{\langle \Omega_T | \Psi_A(x_2, t_2)\Psi_A^\dagger(x_1, t_2)|\Omega_T\rangle}{\langle \Omega_T|\Omega_T\rangle}. \tag{80}$$

This argument depends only on the general saddle-point approximation in the description of the state of equilibrium, and also holds in the case of anyons. The further computation of the field correlator based on equation (80) is similar to the zero-temperature case, the main difference being the change of the measure of integration:

$$\int_{-q}^{q} \mathrm{d}\lambda \to \int_{-\infty}^{\infty} \mathrm{d}\lambda\,\vartheta(\lambda, T, h) \qquad \text{with} \qquad \vartheta(\lambda, T, h) = \frac{1}{1 + \mathrm{e}^{(\lambda^2 - h)/T}}. \tag{81}$$

The final result for the temperature-dependent correlator is then

$$\left\langle \Psi_A(x_2, t_2)\Psi_A^\dagger(x_1, t_1)\right\rangle_T = \mathrm{e}^{\mathrm{i}ht_{21}}\left(\frac{1}{2\pi}G(t_{12}, x_{12}) + \frac{\partial}{\partial\alpha}\right)\det(1 + \hat{V}_T)\bigg|_{\alpha=0}, \tag{82}$$

where the kernel of the integral operator $\hat{V}_T$ is

$$V_T(\lambda, \mu) = \sqrt{\vartheta(\lambda)}V_0(\lambda, \mu)\sqrt{\vartheta(\mu)},$$

$$= \cos^2(\pi\kappa/2)\exp\left\{-\frac{\mathrm{i}}{2}t_{12}(\lambda^2 + \mu^2) + \frac{\mathrm{i}}{2}x_{12}(\lambda + \mu)\right\}\sqrt{\vartheta(\lambda)\vartheta(\mu)}$$

$$\times \left[\frac{E(\lambda|t_{12}, x_{12}) - E(\mu|t_{12}, x_{12})}{\pi^2(\lambda - \mu)} - \frac{\alpha}{2\pi^3}E(\lambda|t_{12}, x_{12})E(\mu|t_{12}, x_{12})\right], \tag{83}$$

and the operator acts on an arbitrary function $f(\mu)$ as

$$(V_T f)(\lambda) = \int_{-\infty}^{\infty} V_T(\lambda, \mu)f(\mu)\,\mathrm{d}\mu. \tag{84}$$

4. Equivalence with the Lenard formula

In the earlier paper [19], we obtained the anyonic generalization of the Lenard formula for the equal-time field correlator or, equivalently, reduced density matrices of anyons. In the case of the first reduced density matrix, the anyonic Lenard formula reads

$$(x|\rho_1^a|x')_\pm = \frac{1}{\pi}\det\left(1 - \gamma\hat{\theta}_T^\pm \bigg|\begin{matrix}x\\x'\end{matrix}\right)\bigg|_{\gamma=(1+\mathrm{e}^{\pm\mathrm{i}\pi\kappa})/\pi}, \tag{85}$$

where the kernel of the integral operators $\hat{\theta}_T^\pm$ is

$$\theta_T(\xi - \eta) = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\frac{\mathrm{e}^{\mathrm{i}(\xi-\eta)\lambda}}{1 + \mathrm{e}^{(\lambda^2 - h)/T}}, \tag{86}$$

and their action on an arbitrary function is defined as

$$\left(\hat{\theta}_T^\pm f\right)(\xi) = \int_{I_\pm}\theta_T(\xi - \eta)f(\eta)\,\mathrm{d}\eta. \tag{87}$$

In these expressions, the plus sign refers to the situation when $x' > x$ and $I_+ = [x, x']$, and the minus sign (–) to the situation when $x' < x$ and $I_- = [x', x]$. The resolvent kernels associated with the kernel $\theta_T(x, y)$ acting on the intervals $I_\pm$ are denoted by $\varrho_T^\pm(\xi, \eta)$ and satisfy the equations:

$$\varrho_T^\pm(\xi, \eta) - \frac{(1 + e^{\pm i\pi\kappa})}{\pi} \int_{I_\pm} \theta_T(\xi - \xi') \varrho_T^\pm(\xi', \eta) \, d\xi' = \theta_T(\xi - \eta). \tag{88}$$

One can rewrite equation (85) in terms of the resolvent kernel ϱ_T and the field correlator as [19]

$$\left\langle \Psi_A^\dagger(x') \Psi_A(x) \right\rangle_{T,\pm} = \frac{1}{\pi} \varrho_T^\pm(x', x) \det\left(1 - \gamma \hat{\theta}_T^\pm\right)\Big|_{\gamma = (1 + e^{\pm i\pi\kappa})/\pi}, \tag{89}$$

where again, the plus sign refers to the case $x' > x$ and the minus sign (–) to $x < x'$. Next, we show that equation (89) is reproduced by the results obtained in the previous section when they are specialized to the equal-time correlators. We treat the two cases, $x' > x$ and $x' < x$, separately.

4.1. The static correlator $\left\langle \Psi_A(-x) \Psi_A^\dagger(x) \right\rangle_T$

Equations (54) and (64) show that in the static case

$$\frac{1}{2\pi} G(0, x) = \delta(x). \tag{90}$$

Using this relation and equations (76) and (83), we see that the equal-time field correlator can be written as

$$\left\langle \Psi_A(-x) \Psi_A^\dagger(x) \right\rangle_T = \left(\delta(2x) + \frac{\partial}{\partial\alpha}\right) \det\left(1 - \frac{(1 + e^{i\pi\kappa})}{\pi} \hat{K}_T + \alpha \frac{e^{i\pi\kappa}}{2\pi} \hat{A}_T^+\right)\Big|_{\alpha=0}, \tag{91}$$

where $\hat{K}_T$ and $\hat{A}_T^+$ are the integral operators acting on the real axis and defined by kernels

$$K_T(\lambda, \mu) = \sqrt{\vartheta(\lambda)} \frac{\sin x(\lambda - \mu)}{\lambda - \mu} \sqrt{\vartheta(\mu)}, \tag{92}$$

and

$$A_T^+(\lambda, \mu) = \sqrt{\vartheta(\lambda)} \, e^{-ix(\lambda+\mu)} \sqrt{\vartheta(\mu)}. \tag{93}$$

At zero temperature, both operators act on the interval $[-q, q]$ and their kernels are

$$K(\lambda, \mu) = \frac{\sin x(\lambda - \mu)}{\lambda - \mu}, \qquad A^+(\lambda, \mu) = e^{-ix(\lambda+\mu)}. \tag{94}$$

The commutation relation (13) shows that

$$\left\langle \Psi_A(-x) \Psi_A^\dagger(x) \right\rangle_T = e^{i\pi\kappa} \left\langle \Psi_A^\dagger(x) \Psi_A(-x) \right\rangle_T + \delta(2x). \tag{95}$$

This means that in order to prove the equivalence with the Lenard formula, we have to show that

$$G^+(\kappa, x, T) \equiv \frac{\partial}{\partial\alpha} \det\left(1 - \frac{(1 + e^{i\pi\kappa})}{\pi} \hat{K}_T + \alpha \frac{e^{i\pi\kappa}}{2\pi} \hat{A}_T^+\right)\Big|_{\alpha=0} = e^{i\pi\kappa} \left\langle \Psi_A^\dagger(x) \Psi_A(-x) \right\rangle_T, \tag{96}$$

where $\left\langle \Psi_A^\dagger(x) \Psi_A(-x) \right\rangle_T$ is given by (89). For a general integral operator with kernel V, one of the useful expressions for the Fredholm determinant is

$$\ln \det(1 - \gamma \hat{V}) = -\sum_{n=1}^\infty \frac{\gamma^n}{n} \operatorname{Tr} V^n.$$

Making use of this formula, we obtain

$$G^+(\kappa, x, T) = \frac{e^{i\pi\kappa}}{2\pi} \text{Tr} \left[(1 - \gamma \hat{K}_T)^{-1} \hat{A}_T^+ \right] \det(1 - \gamma \hat{K}_T)|_{\gamma=(1+e^{i\pi\kappa})/\pi}. \tag{97}$$

Denoting as $f_-^+(\lambda)$ the solution of the integral equation

$$f_-^+(\lambda) - \frac{(1 + e^{i\pi\kappa})}{\pi} \int_{-\infty}^{\infty} K_T(\lambda, \mu) f_-^+(\mu) \, d\mu = \sqrt{\vartheta(\lambda)} \, e^{-ix\lambda}, \tag{98}$$

we can rewrite (97) as

$$G^+(\kappa, x, T) = \frac{e^{i\pi\kappa}}{2\pi} \int_{-\infty}^{\infty} e^{-ix\lambda} f_-^+(\lambda) \sqrt{\vartheta(\lambda)} \, d\lambda \, \det(1 - \gamma \hat{K}_T)|_{\gamma=(1+e^{i\pi\kappa})/\pi}. \tag{99}$$

We will now show that

$$\det(1 - \gamma \hat{K}_T) = \det\left(1 - \gamma \hat{\theta}_T^+\right), \tag{100}$$

where the operator $\hat{\theta}_T$ is described by equations (86) and (87), and $\gamma = (1 + e^{i\pi\kappa})/\pi$. Direct and inverse Fourier transforms of a function g can be defined to include as integration measure $\sqrt{\vartheta(\lambda)}$:

$$\tilde{g}(\lambda) = \frac{1}{2\pi\sqrt{\vartheta(\lambda)}} \int_{-\infty}^{\infty} d\xi \, e^{i\lambda\xi} g(\xi), \qquad g(\xi) = \int_{-\infty}^{\infty} d\lambda \sqrt{\vartheta(\lambda)} \, e^{-i\lambda\xi} \tilde{g}(\lambda). \tag{101}$$

With this definition, taking the Fourier transform of the integral equation

$$g(\xi) - \gamma \int_{-x}^{x} \theta_T(\xi - \xi') g(\xi') \, d\xi' = G(\xi), \tag{102}$$

we obtain

$$\tilde{g}(\lambda) - \gamma \int_{-\infty}^{\infty} K_T(\lambda - \mu) \tilde{g}(\mu) \, d\mu = \tilde{G}(\lambda). \tag{103}$$

Coincidence of the two equations implies the equality (100) of the determinants.

The final step in proving the equivalence of equations (89) and (91) is to show that

$$\varrho_T^+(x, -x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-ix\lambda} f_-^+(\lambda) \sqrt{\vartheta(\lambda)} \, d\lambda. \tag{104}$$

The Fourier transform of the equation defining the resolvent kernel ϱ_T

$$\varrho_T^+(\xi, -x) - \frac{(1 + e^{i\pi\kappa})}{\pi} \int_{-x}^{x} \theta_T(\xi - \xi') \varrho_T^+(\xi', -x) \, d\xi' = \theta_T(\xi + x) \tag{105}$$

gives

$$\tilde{\varrho}_T^+(\lambda, -x) - \frac{(1 + e^{i\pi\kappa})}{\pi} \int_{-\infty}^{\infty} K_T(\lambda - \mu) \tilde{\varrho}_T^+(\mu, -x) \, d\mu = \frac{1}{2} e^{-ix\lambda} \sqrt{\vartheta(\lambda)}. \tag{106}$$

Comparison of this equation with the definition of $f_-^+(\lambda)$ (98) shows that

$$\tilde{\varrho}_T^+(\lambda, -x) = \tfrac{1}{2} f_-^+(\lambda). \tag{107}$$

Taking the inverse Fourier transform of (107) proves (104). Thus, we have shown that for $x' > x$, the Lenard formula (89) is equivalent to the result (91) for the static field correlator that follows from the direct summation of the form factors.

14

4.2. The static correlator $\left\langle \Psi_A(x)\Psi_A^\dagger(-x) \right\rangle_T$

In this case, the proof of the equivalence of the two approaches is very similar to what was just discussed for $x' > x$. Equations (78) and (83) show that the static field correlator is

$$\left\langle \Psi_A(x)\Psi_A^\dagger(-x) \right\rangle_T = \left(\delta(2x) + \frac{\partial}{\partial\alpha} \right) \det\left(1 - \frac{(1+\mathrm{e}^{-\mathrm{i}\pi\kappa})}{\pi}\hat{K}_T + \alpha\frac{\mathrm{e}^{-\mathrm{i}\pi\kappa}}{2\pi}\hat{A}_T^- \right)\bigg|_{\alpha=0}, \tag{108}$$

where $\hat{K}_T$ is given by (92) and

$$A_T^-(\lambda,\mu) = \sqrt{\vartheta(\lambda)}\,\mathrm{e}^{\mathrm{i}x(\lambda+\mu)}\sqrt{\vartheta(\mu)}. \tag{109}$$

From the commutation relation (13) we see that

$$\left\langle \Psi_A(x)\Psi_A^\dagger(-x) \right\rangle_T = \mathrm{e}^{-\mathrm{i}\pi\kappa}\left\langle \Psi_A^\dagger(-x)\Psi_A(x) \right\rangle_T + \delta(2x), \tag{110}$$

so we have to show that

$$G^-(\kappa,x,T) \equiv \frac{\partial}{\partial\alpha}\det\left(1 - \frac{(1+\mathrm{e}^{-\mathrm{i}\pi\kappa})}{\pi}\hat{K}_T + \alpha\frac{\mathrm{e}^{-\mathrm{i}\pi\kappa}}{2\pi}\hat{A}_T^- \right)\bigg|_{\alpha=0}$$

$$= \mathrm{e}^{-\mathrm{i}\pi\kappa}\left\langle \Psi_A^\dagger(-x)\Psi_A(x) \right\rangle_T \tag{111}$$

where $\left\langle \Psi_A^\dagger(-x)\Psi_A(x) \right\rangle_T$ is given by equation (89). Similarly to the discussion in the previous section, we can rewrite G^- as

$$G^-(\kappa,x,T) = \frac{\mathrm{e}^{-\mathrm{i}\pi\kappa}}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{+\mathrm{i}x\lambda}f_+^-(\lambda)\sqrt{\vartheta(\lambda)}\,\mathrm{d}\lambda\,\det(1-\gamma\hat{K}_T)|_{\gamma=(1+\mathrm{e}^{-\mathrm{i}\pi\kappa})/\pi}, \tag{112}$$

where $f_+^-(\lambda)$ is the solution of the integral equation

$$f_+^-(\lambda) - \frac{(1+\mathrm{e}^{-\mathrm{i}\pi\kappa})}{\pi}\int_{-\infty}^{\infty}K_T(\lambda,\mu)f_+^-(\mu)\,\mathrm{d}\mu = \sqrt{\vartheta(\lambda)}\,\mathrm{e}^{+\mathrm{i}x\lambda}. \tag{113}$$

The equality of the Fredholm determinants of the operators $\hat{K}_T$ and $\hat{\theta}_T$ was shown in the previous Section, so it remains to prove that

$$\varrho_T^-(-x,x) = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{e}^{+\mathrm{i}x\lambda}f_+^-(\lambda)\sqrt{\vartheta(\lambda)}\,\mathrm{d}\lambda. \tag{114}$$

Again, taking the Fourier transform of

$$\varrho_T^-(\xi,x) - \frac{(1+\mathrm{e}^{-\mathrm{i}\pi\kappa})}{\pi}\int_{-x}^{x}\theta_T(\xi-\xi')\varrho_T^+(\xi',x)\,\mathrm{d}\xi' = \theta_T(\xi-x), \tag{115}$$

we obtain

$$\tilde{\varrho}_T^-(\lambda,x) - \frac{(1+\mathrm{e}^{-\mathrm{i}\pi\kappa})}{\pi}\int_{-\infty}^{\infty}K_T(\lambda-\mu)\tilde{\varrho}_T^-(\mu,x)\,\mathrm{d}\mu = \frac{1}{2}\mathrm{e}^{+\mathrm{i}x\lambda}\sqrt{\vartheta(\lambda)}, \tag{116}$$

which shows that

$$\tilde{\varrho}_T^-(\lambda,x) = \tfrac{1}{2}f_+^-(\lambda). \tag{117}$$

The inverse Fourier transform of (117) gives the correct result (114).

5. Conclusions

In summary, we have obtained the time- and temperature-dependent correlation functions of fields for impenetrable 1D anyons as Fredholm determinants. The Fourier transform of the corresponding integral equations proves the equivalence of our approach with the anyonic Lenard formula derived previously (equation (57) of [19]) for the one-particle reduced density matrix of anyons. The same technique can be used to obtain the multi-point correlation functions from the Lenard formula for *n*-particle reduced density matrices (equation (56) of [19]). The next step in the exact calculation of the anyonic correlation functions is to use the determinant representation derived in this work to obtain a classical integrable system of nonlinear differential equations characterizing these functions. These equations should make it possible to construct the short-distance and low-density expansions for the correlators. This will be addressed in a future publication.

Acknowledgments

Appendix A. Anyonic form factors

In this appendix, we prove equation (36). Consider first the simple example of the form factor $F_{3,2}$:

$$F_{3,2}(x) = \frac{1}{2\sqrt{3}} \int d^3y \, d^2z \, \chi_3^*(y_1, y_2, y_3) \chi_2(z_1, z_2) \langle 0|\Psi_A(y_1)\Psi_A(y_2)$$
$$\times \Psi_A(y_3)\Psi_A^\dagger(x)\Psi_A^\dagger(z_2)\Psi_A^\dagger(z_1)|0\rangle. \tag{A.1}$$

If one defines

$$A = \langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A(y_3)\Psi_A^\dagger(x)\Psi_A^\dagger(z_2)\Psi_A^\dagger(z_1)|0\rangle, \tag{A.2}$$

then successive applications of the commutation relation (13) followed by the equation (16) give

$$A = \langle 0|\Psi_A(y_1)\Psi_A(y_2)\left[\Psi_A^\dagger(x)\Psi_A(y_3)\,e^{-i\pi\kappa\epsilon(y_3-x)} + \delta(y_3-x)\right]\Psi_A^\dagger(z_2)\Psi_A^\dagger(z_1)|0\rangle$$
$$= \langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(x)\left[\Psi_A^\dagger(z_2)\Psi_A(y_3)\,e^{-i\pi\kappa\epsilon(y_3-z_2)} + \delta(y_3-z_2)\right]$$
$$\times \Psi_A^\dagger(z_1)|0\rangle\,e^{-i\pi\kappa\epsilon(y_3-x)} + \langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(z_2)\Psi_A^\dagger(z_1)|0\rangle\delta(y_3-x)$$
$$= \underbrace{\langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(x)\Psi_A^\dagger(z_2)|0\rangle\delta(y_3-z_1)\,e^{-i\pi\kappa[\epsilon(y_3-z_2)+\epsilon(y_3-x)]}}_{\textbf{(a)}}$$
$$+ \underbrace{\langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(x)\Psi_A^\dagger(z_1)|0\rangle\delta(y_3-z_2)\,e^{-i\pi\kappa\epsilon(y_3-x)}}_{\textbf{(b)}}$$
$$+ \underbrace{\langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(z_2)\Psi_A^\dagger(z_1)|0\rangle\delta(y_3-x)}_{\textbf{(c)}}. \tag{A.3}$$

Performing similar transformations, we obtain

$$\mathbf{a} = \delta(y_1-x)\delta(y_2-z_2)\delta(y_3-z_1)\,e^{-i\pi\kappa[\epsilon(y_2-x)+\epsilon(y_3-z_2)+\epsilon(y_3-x)]}$$
$$+ \delta(y_1-z_2)\delta(y_2-x)\delta(y_3-z_1)\,e^{-i\pi\kappa[\epsilon(y_3-z_2)+\epsilon(y_3-x)]}, \tag{A.4}$$

$$\mathbf{b} = \delta(y_1 - x)\delta(y_2 - z_1)\delta(y_3 - z_2)\,\mathrm{e}^{-\mathrm{i}\pi\kappa[\epsilon(y_2-x)+\epsilon(y_3-x)]}$$
$$+ \delta(y_1 - z_1)\delta(y_2 - x)\delta(y_3 - z_2)\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(y_3-x)}, \tag{A.5}$$

$$\mathbf{c} = \delta(y_1 - z_2)\delta(y_2 - z_1)\delta(y_3 - x)\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(y_2-z_2)} + \delta(y_1 - z_1)\delta(y_2 - z_2)\delta(y_3 - z_3). \tag{A.6}$$

Substituting $A = \mathbf{a} + \mathbf{b} + \mathbf{c}$ into (A.1), we have for the form factor

$$F_{3,2}(x) = \frac{1}{2\sqrt{3}} \int \mathrm{d}^2 z \{\chi_3^*(x, z_2, z_1)\chi_2(z_1, z_2)\,\mathrm{e}^{-\mathrm{i}\pi\kappa[\epsilon(z_2-x)+\epsilon(z_1-z_2)+\epsilon(z_1-x)]}$$
$$+ \chi_3^*(z_2, x, z_1)\chi_2(z_1, z_2)\,\mathrm{e}^{-\mathrm{i}\pi\kappa[\epsilon(z_1-z_2)+\epsilon(z_1-x)]}$$
$$+ \chi_3^*(x, z_1, z_2)\chi_2(z_1, z_2)\,\mathrm{e}^{-\mathrm{i}\pi\kappa[\epsilon(z_1-x)+\epsilon(z_2-x)]}$$
$$+ \chi_3^*(z_1, x, z_2)\chi_2(z_1, z_2)\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(z_2-x)} + \chi_3^*(z_2, z_1, x)\chi_2(z_1, z_2)\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(z_1-z_2)}$$
$$+ \chi_3^*(z_1, z_2, x)\chi_2(z_1, z_2)\}. \tag{A.7}$$

Using the anyonic property (18) of the wavefunctions, and its complex conjugate:

$$\chi^*(\ldots, z_i, z_{i+1}, \ldots) = \mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(z_i-z_{i+1})}\chi^*(\ldots, z_{i+1}, z_i, \ldots), \tag{A.8}$$

we reduce equation (A.7) to the final expression for the form factor

$$F_{3,2}(x) = \sqrt{3} \int \mathrm{d}^2 z \,\chi_3^*(z_1, z_2, x)\chi_2(z_1, z_2). \tag{A.9}$$

The calculations leading to equation (A.9) can be generalized to arbitrary N:

$$F_{N+1,N}(x) = \langle \Psi_{N+1} | \Psi_A^\dagger(x) | \Psi_N \rangle = \sqrt{N+1} \int \mathrm{d}^N z \,\chi_{N+1}^*(z_1, \ldots, z_N, x)\chi_N(z_1, \ldots, z_N). \tag{A.10}$$

This result follows from equation (35) by noting that the statistical phase factors in the commutation relations (13)–(15) of the field operators are compensated by the exchange property (18) of the wavefunctions. This means that the pairing of the $\Psi_A^\dagger(x)$ operator with any of the $\Psi_A(y_j)$ operators produces $N + 1$ identical terms in which the coordinate x is made the last coordinate of the wavefunction χ_{N+1}. After that, the integrals over z's and remaining y's can be limited to the ordered regions $z_1 > z_2 >, \ldots, > z_N$ and $y_1 > y_2 >, \ldots, > y_N$ giving directly (A.10).

Appendix B. The thermodynamic limit of singular sums

In this appendix, we study the behavior of the functions defined by equations (61), (62) and (63) in the thermodynamic limit of large length L of normalization interval. We start with (61). In this case, the function summed over the momenta λ is sufficiently smooth, so that the anyonic shift $2\pi\delta'/L$ of the momenta becomes negligible when $L \to \infty$, and one can pass directly from the sum to the integral over λ:

$$G(t, x) \equiv \lim_{L\to\infty} G_L(t, x) = \frac{2\pi}{L} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} e(\lambda_j | t, x) = \int_{-\infty}^{\infty} e(\lambda | t, x)\,\mathrm{d}\lambda. \tag{B.1}$$

The regularization $t \to t + \mathrm{i}0$ for $e(\lambda | t, x) = \exp(\mathrm{i}t\lambda^2 - \mathrm{i}x\lambda)$ is implied in these expressions.

Next, we turn to equation (62). In this case, the function under the sum is no longer smooth in the thermodynamic limit. We transform it by separating the singular part that can

17

be summed explicitly:

$$
E(\mu_k|t,x) \equiv \lim_{L\to\infty} E_L(\mu_k|t,x) = \frac{2\pi}{L} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x)}{\lambda_j - \mu_k}
$$

$$
= \frac{2\pi}{L} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{\lambda_j - \mu_k} + e(\mu_k|t,x) \sum_{n=-\infty}^{\infty} \left(n - \frac{\kappa+1}{2}\right)^{-1}.
$$

(B.2)

In the last line here we have used equation (39). The first term in (B.2) is now a smooth function, so as before, we can directly replace the sum with the integral, since the anyonic shift of the momenta does not affect the value of the integral. The integral can then be transformed as follows

$$
\int_{-\infty}^{\infty} d\lambda \frac{e(\lambda|t,x) - e(\mu_k|t,x)}{\lambda - \mu_k} = \text{P.V.} \int_{-\infty}^{\infty} d\lambda \frac{e(\lambda|t,x)}{\lambda - \mu_k} - e(\mu_k|t,x)\text{P.V.} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda - \mu_k}
$$

$$
= \text{P.V.} \int_{-\infty}^{\infty} d\lambda \frac{e(\lambda|t,x)}{\lambda - \mu_k}.
$$

(B.3)

Under the natural interpretation of the sum in the second term in (B.2), it can be simplified using formula 1.421.(3) of [21], $\pi\cot(\pi x) = (1/x) + 2x\sum_{n=1}^{\infty}(x^2 - n^2)^{-1}$:

$$
\sum_{n=-\infty}^{\infty} \left(n - \frac{\kappa+1}{2}\right)^{-1} = \pi\tan\left(\frac{\pi\kappa}{2}\right).
$$

(B.4)

Collecting the two terms we finally get

$$
E(\mu_k|t,x) = \text{P.V.} \int_{-\infty}^{\infty} d\lambda \frac{e(\lambda|t,x)}{\lambda - \mu_k} + e(\mu_k|t,x)\pi\tan\left(\frac{\pi\kappa}{2}\right).
$$

(B.5)

The function defined by equation (63) is more singular than $E(\mu_k|t,x)$ ((B.2). To transform it, we use the same strategy of separating the most divergent terms that can be summed explicitly:

$$
\tilde{E}(\mu_k|t,x) \equiv \lim_{L\to\infty} \tilde{E}_L(\mu_k|t,x) = \frac{4}{L^2}\cos^2(\pi\kappa/2) \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x)}{(\lambda_j - \mu_k)^2},
$$

$$
= \frac{4}{L^2}\cos^2(\pi\kappa/2)\Bigg(\sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{(\lambda_j - \mu_k)^2}
$$

$$
+ e(\mu_k|t,x)\frac{L^2}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{\left(n - \frac{\kappa+1}{2}\right)^2}\Bigg).
$$

(B.6)

Defining

$$
f(\mu_k) = \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{\lambda_j - \mu_k},
$$

(B.7)

one has

$$
\sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{(\lambda_j - \mu_k)^2} = \frac{\partial f(\mu_k)}{\partial \mu_k} + \frac{\partial e(\mu_k|t,x)}{\partial \mu_k} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{1}{\lambda_j - \mu_k}.
$$

(B.8)

Taking the limit $L \to \infty$ and using (B.3) and (B.4) in this equation we obtain

$$\lim_{L \to \infty} \frac{L}{2\pi} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{(\lambda_j - \mu_k)^2}$$

$$= \frac{\partial}{\partial \mu_k} \left(\text{P.V.} \int_{-\infty}^{\infty} d\lambda \frac{e(\lambda|t,x)}{\lambda - \mu_k} \right) + \frac{\partial e(\mu_k|t,x)}{\partial \mu_k} \pi \tan\left(\frac{\pi\kappa}{2}\right). \tag{B.9}$$

For the second term on the RHS of (B.6) we use the formula 1.422.(4) of [21] $\pi^2/\sin^2(\pi x) = \sum_{n=-\infty}^{\infty}(n-x)^{-2}$ to get

$$\sum_{n=-\infty}^{\infty} \left(n - \frac{\kappa+1}{2} \right)^{-2} = \frac{\pi^2}{\cos^2(\pi\kappa/2)}. \tag{B.10}$$

Collecting all the terms we have the final result

$$\tilde{E}(\mu_k|t,x) = e(\mu_k|t,x) + \frac{2\cos^2(\pi\kappa/2)}{\pi L} \frac{\partial e(\mu_k|t,x)}{\partial \mu_k} \pi \tan\left(\frac{\pi\kappa}{2}\right)$$

$$+ \frac{2\cos^2(\pi\kappa/2)}{\pi L} \frac{\partial}{\partial \mu_k} \left(\text{P.V.} \int_{-\infty}^{\infty} d\lambda \frac{e(\lambda|t,x)}{\lambda - \mu_k} \right).. \tag{B.11}$$

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